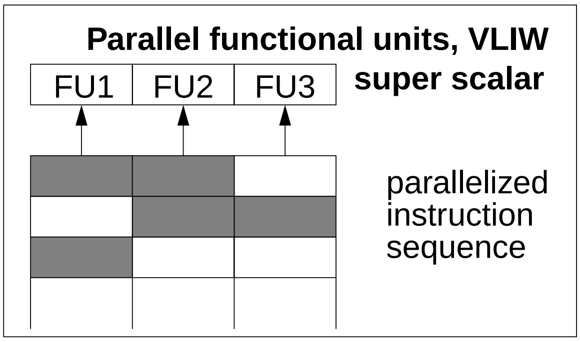
5 Code Parallelization

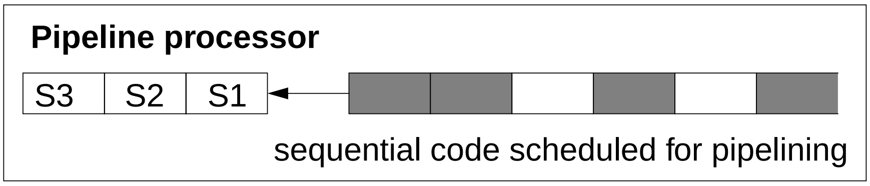
Processor with **instruction level parallelism (ILP)**
executes several instructions in parallel.

Classes of processors and parallelism:
VLIW, super scalar
Pipelined processors
Data parallel processors



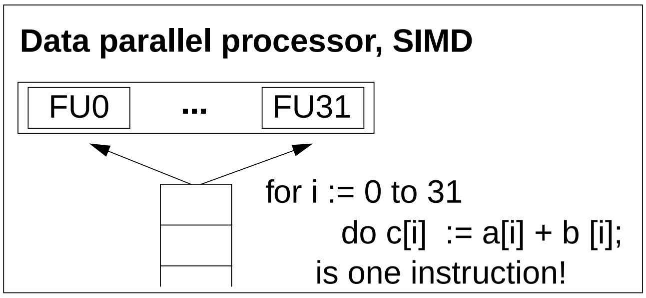
Compiler **analyzes sequential programs** to **exhibit potential parallelism** on instruction level;

model **dependences between computations**



Compiler arranges instructions for shortest execution time:
instruction scheduling

Compiler **analyzes loops** to execute them in parallel
loop transformation
array transformation



Lecture Compilation Methods SS 2013 / Slide 501

Objectives:

3 abstractions of processor parallism

In the lecture:

- explain the abstract models
- relate to real processors
- explain the instruction scheduling tasks

Suggested reading:

Kastens / Übersetzerbau, Section 8.5

Questions:

- What has to be known about instruction execution in order to solve the instruction scheduling problem in the compiler?

5.1 Instruction Scheduling
Data Dependence Graph

Exhibit potential **fine-grained parallelism** among operations.
Sequential code is over-specified!

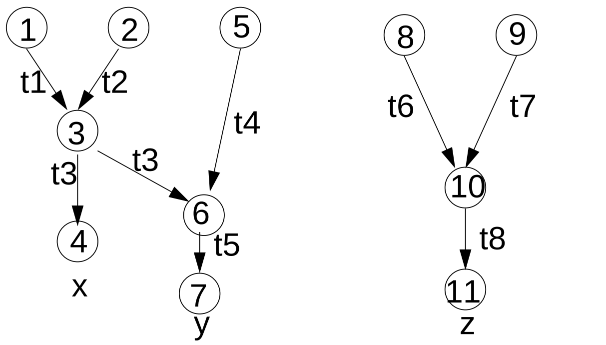
Data dependence graph (DDG) for a basic block:

- **Node**: operation;
- **Edge** a -> b: operation b uses the result of operation a

Example for a basic block:

```
 1:  t1   := a
 2:  t2   := b
 3:  t3   := t1 + t2
 4:  x    := t3
 5:  t4   := c
 6:  t5   := t3 + t4
 7:  y    := t5
 8:  t6   := d
 9:  t7   := e
10:  t8   := t6 + t7
11:  z    := t8
```

data dependence graph



ti are symbolic registers, store intermediate results, obey single assignment rule

Lecture Compilation Methods SS 2013 / Slide 502

Objectives:

DDG exhibits parallelism

In the lecture:

- Show where sequential code is overspecified.
- Derive reordered sequences from the ddg.
- single assignment for ti: ti contains exactly one value; ti is not reused for other values.
- Without that assumption further dependencies have to manifest the order of assignments to those registers.

Suggested reading:

Kastens / Übersetzerbau, Section 8.5, Abb. 8.5-1

Assignments:

- Write the operations of the basic block in a different order, such that the effect is not changed and the same DDG is produced.

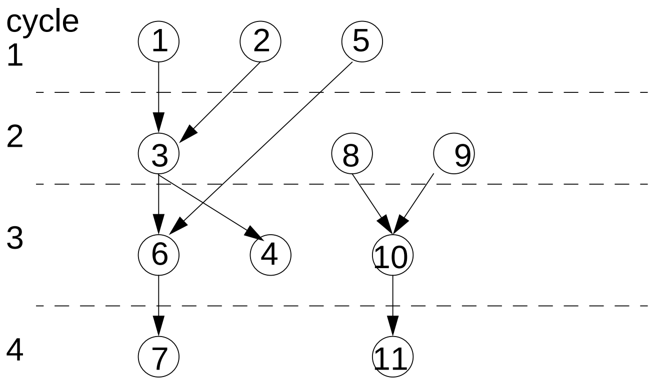
Questions:

- Why does this example have so much freedom for rearranging operations?
- Why are further dependences necessary if registers are allocated?

List Scheduling

Input: data dependence graph
Output: a schedule of **at most k operations per cycle**,
such that all **dependences point forward;** DDG arranged in levels

Algorithm: A **ready list** contains all operations that are **not yet scheduled**,
but whose **predecessors are scheduled**
Iterate: **select** from the ready list up to k operations for the next cycle (heuristic),
update the ready list



- Algorithm is **optimal** only for **trees.**
- **Heuristic**: Keep ready list sorted by distance to an end node, e. g.

  (1 2 5) (8 9 3) (6 10 4) (7 11)

  without this heuristic:
  (1 8 9) (2 5 10) (3 11) (6 4) (7)

  ( ) operations in one cycle

Critical paths determine minimal schedule length: e. g. 1 -> 3 -> 6 -> 7

Lecture Compilation Methods SS 2013 / Slide 503

Objectives:

A simple fundamental scheduling algorithm

In the lecture:

- Explain the algorithm using the example.
- Show variants of orders in the ready list, and their consequences.
- Explain the heuristic.

Suggested reading:

Kastens / Übersetzerbau, Section 8.5.1

Assignments:

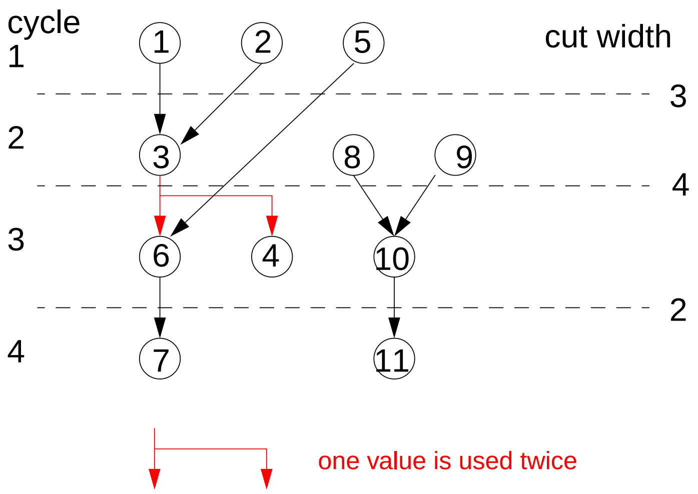
- Write the parallel code for this example.

Questions:

- Explain the heuristic with respect to critical paths.

Variants and Restrictions for List Scheduling

- Allocate **as soon as possible**, ASAP (C-5.3); as **late** as possible, ALAP
- Operations have **unit execution time** (C-5.3); **different execution times:** selection avoids conflicts with already allocated operations
- Operations only on **specific functional units** (e. g. 2 int FUs, 2 float FUs)
- **Resource restrictions** between operations, e. g. <= 1 load or store per cycle



Scheduled DDG models **number of needed registers**:

- arc represents the use of an intermediate result
- **cut width** through a level gives the number of **registers needed**

The tighter the schedule the more registers are needed (*register pressure*).

Lecture Compilation Methods SS 2013 / Slide 504

Objectives:

A simple fundamental scheduling algorithm

In the lecture:

- Explain ASAP and ALAP.
- Explain restrictions on the selection of operations.
- Show how the register need is modeled.

Suggested reading:

Kastens / Übersetzerbau, Section 8.5.1

Assignments:

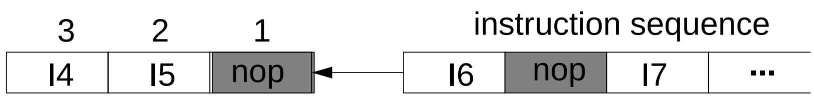
- The algorithm allocates an operation as soon as possible (ASAP). Describe a variant of the algorithm which allocates an operation as late as possible (ALAP).
- Describe a variant, that allocates operations of different execution times.

Questions:

- Compare the way register need is modeled with the approach of Belady for register allocation.
- Why need tight schedules more registers?

Instruction Scheduling for Pipelining

Instruction pipeline with 3 stages:



Dependent instructions may not follow one another immediately.

Schedule rearranges the operation sequence, to minimize the number of delays:

without scheduling:

```
1:   t1   := a
2:   t2   := b
     nop
3:   t3   := t1 + t2
     nop
4:   x    := t3
5:   t4   := c
     nop
6:   t5   := t3 + t4
     nop
7:   y    := t5
8:   t6   := d
9:   t7   := e
     nop
10:  t8   := t6 + t7
     nop
11:  z    := t8
```

with scheduling

no delays

```
1:   t1   := a
2:   t2   := b
5:   t4   := c
3:   t3   := t1 + t2
8:   t6   := d
9:   t7   := e
6:   t5   := t3 + t4
10:  t8   := t6 + t7
4:   x    := t3
7:   y    := t5
11:  z    := t8
```


Lecture Compilation Methods SS 2013 / Slide 505

Objectives:

Restrictions for pipelining

In the lecture:

- Requirements of pipelining processors.
- Compiler reorders to meet the requirements, inserts nops (empty operations), if necessary.
- Some processors accept too close operations, delays the second one by a hardware interlock.
- Hardware bypasses may relax the requirements

Suggested reading:

Kastens / Übersetzerbau, Section 8.5.2

Questions:

- Why are no nops needed in this example?

Instruction Scheduling Algorithm for Pipelining

Algorithm: modified list scheduling:

Select from the ready list such that the selected operation

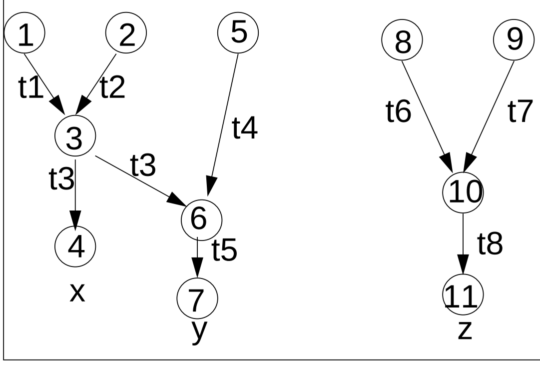
- has a sufficient **distance to all predecessors** in DDG
- has **many successors** (heuristic)
- has a **long path to the end** node (heuristic)

Insert an empty operation if none is selectable.

Ready list with additional information:

opr.	1	2	5	8	9	3	6	4	10	7	11
succ #	1	1	1	1	1	2	1	0	1	0	0
to end	3	3	2	2	2	2	1	1	1	0	0
sched. cycle	1	2	3	5	6	4	7	9	8	10	11

data dependence graph



cycle				
1	1:	t1	:= a	
2	2:	t2	:= b	
3	5:	t4	:= c	
4	3:	t3	:= t1 + t2	**with scheduling**
5	8:	t6	:= d	
6	9:	t7	:= e	
7	6:	t5	:= t3 + t4	
8	10:	t8	:= t6 + t7	
9	4:	x	:= t3	
10	7:	y	:= t5	
11	11:	z	:= t8	

Lecture Compilation Methods SS 2013 / Slide 506

Objectives:

Adapted list scheduling

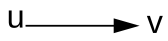
In the lecture:

- Explain the algorithm using the example.
- Explain the selection criteria.

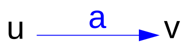
Suggested reading:

Kastens / Übersetzerbau, Section 8.5.2

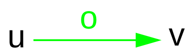
Reused registers: anti- and output-dependences



flow-dependence:
u writes before v uses

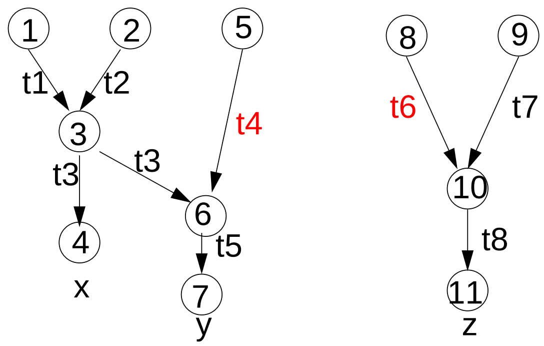


anti-dependence:
u uses a value
before v overwrites it

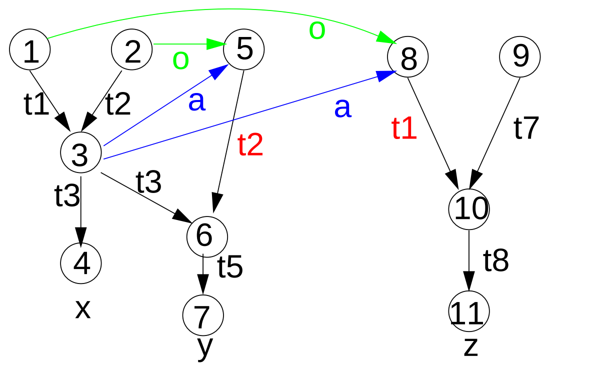


output-dependence:
u writes before v overwrites

DDG with symbolic registers ti
flow-dependences only



DDG with reused registers ti
flow, anti-, and output-dependences



Lecture Compilation Methods SS 2013 / Slide 506b

Objectives:

Understand anti- and output-dependences

In the lecture:

Explain anti- and output-dependences:

- Reuse of registers introduces new dependences

DDG with Loop Carried Dependences

Factorial computation:

program:

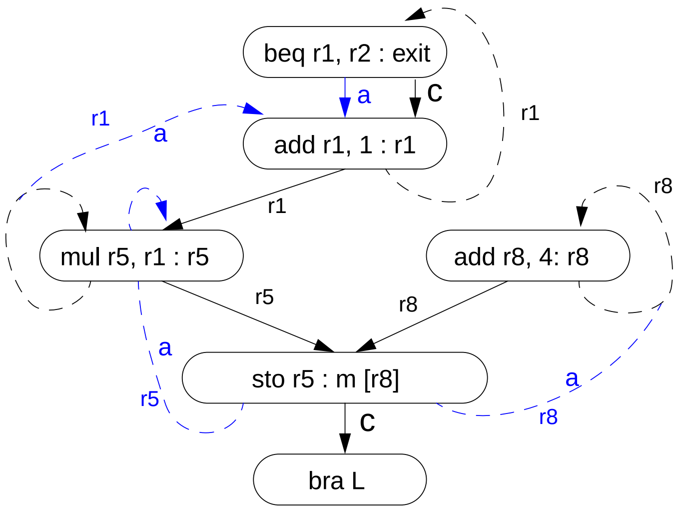
```
i = 0; f = 1;
while ( i != n)
{   i = i + 1;
    f = f * i;

    m[i] = f;
}
```

seq. machine code:

```
L:  beq  r1, r2 : exit
    add  r1, 1  : r1
    mul  r5, r1 : r5
    add  r8, 4  : r8
    sto  r5 : m[r8]
    bra  L
```

Data dependence graph:



u → v **flow-dependence**: u writes before v uses

u - - → v **flow-dependence** into subsequent iteration

u - a → v **anti-dependence**: u uses a value before v overwrites it

u - o → v **output-dependence**: u writes before v overwrites

u c → v **control-dependence**: u has to be executed before v (u or v may branch)

Lecture Compilation Methods SS 2013 / Slide 506d

Objectives:

Loop carried dependences

In the lecture:

Explain loop carried dependences

- the 4 kinds,
- they occur, because a new value is stored in the same register on every iteration,
- they are relevant, because we are going to merge operations of several iterations.

Questions:

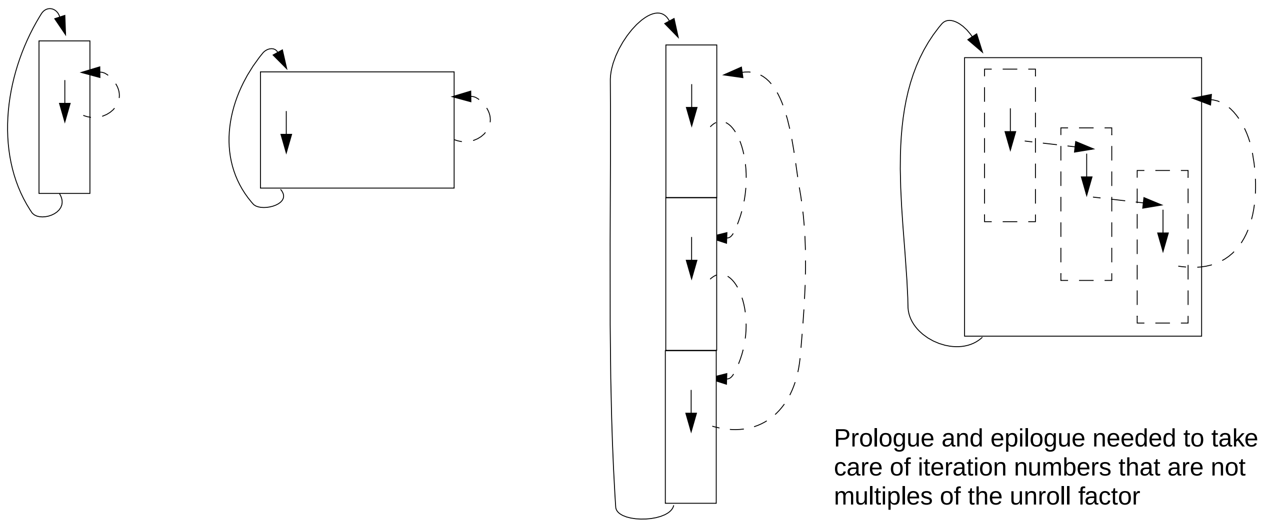
- Explain why loops with arrays can have dependences into later iterations that are not the next one. Give an example.

Loop unrolling

Loop unrolling: A technique for parallelization of loops.

A single loop body does not exhibit enough parallelism => sparse schedule.
Schedule the code (copies) of several adjacent iterations together
=> more compact schedule

sequential loop | parallel schedule for single body | unrolled loop (3 times) | parallel schedule for unrolled loop



Prologue and epilogue needed to take care of iteration numbers that are not multiples of the unroll factor

Lecture Compilation Methods SS 2013 / Slide 506u

Objectives:

Understand the idea of loop unrolling

In the lecture:

- Compare the single body schedule to the schedule of the unrolled loop.
- Explain the consequences of loop carried dependences.

Suggested reading:

Kastens / Übersetzerbau, Section 8.5.2

Software Pipelining

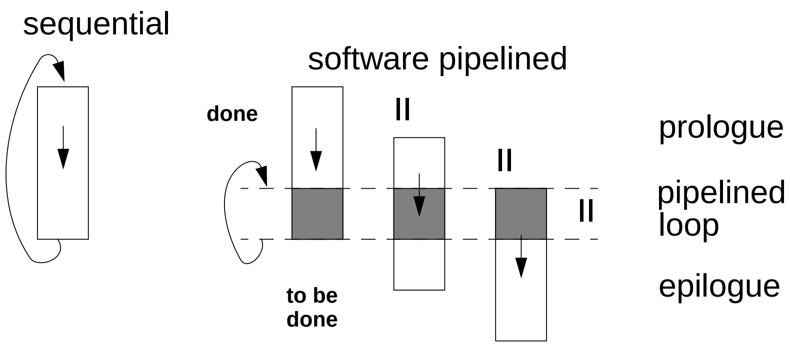
Software Pipelining: A technique for parallelization of loops.

A single loop body does not exhibit enough parallelism => sparse schedule.
Overlap the execution of several adjacent iterations => compact schedule

The **pipelined loop body**

has **each operation** of the original sequential body,
they belong to **several iterations**,
they are **tightly scheduled,**
its length is the **initiation interval II**,
is **shorter** than the original body.

Prologue, epilogue: initiation and finalization code



Lecture Compilation Methods SS 2013 / Slide 507

Objectives:

Understand the underlying idea

In the lecture:

- Explain the underlying idea
- II is both: length of the piplined loop and time between the start of two successive iterations.

Questions:

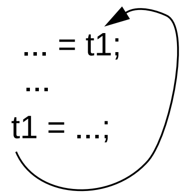
Explain:

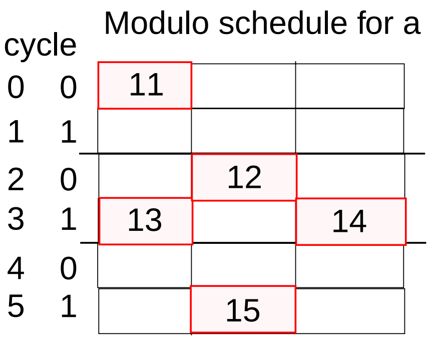
- The shorter the initiation interval is, the greater is the parallelism, and the compacter is the schedule.

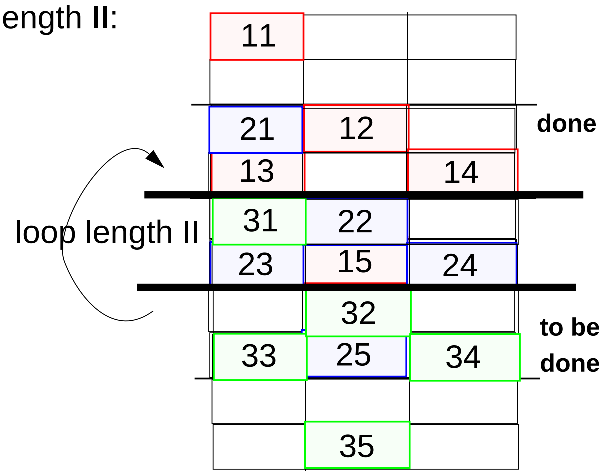
Transform Loops by Software Pipelining

Technique:

1. **Data dependence graph** for the loop body, include **loop carried dependences**.
2. Chose a **small initiation interval II** - not smaller than #instructions / #FUs
3. Make a **„Modulo Schedule“** s for the loop body: Two instructions can not be scheduled on the same FU, i_1 in cycle c_1 and i_2 in cycle c_2, if c_1 mod II = c_2 mod II
4. If (3) does not succeed without conflict, increase II and repeat from 3
5. Allocate the instructions of s in the new loop of length II: i_j scheduled in cycle c_j is allocated to c_j mod II
6. Construct prologue and epilogue.







Lecture Compilation Methods SS 2013 / Slide 508

Objectives:

Understand the technique

In the lecture:

- Explain the algorithm.
- Explain reasons for conflicts in step 4.

Questions:

Explain:

- The shorter the initiation interval is, the greater is the parallelism, and the compacter is the schedule.
- The transformed loop contains each instruction of the loop body exactly once.

Result of Software Pipelining

t	t_m		ADD	MUL	MEM	CTR
0	0	L:				beq r1, r2:exit
1	1		add r1, 1: r1			
2	0		add r8, 4 : r8	mul r5, r1 : r5		
3	1			... mul		
4	0				sto r5 : m r8	
5	1				... sto	
6	0					
7	1					bra L

4 dedicated FUs schedule of the loop body for II = 2

mul and sto need 2 cycles

add and sto in t_m=0, sto reads r8 before add writes it

bra not in cycle 6, it collides with beq: t_m=0

t	t_m		ADD	MUL	MEM	CTR
0	0					beq r1;r2:exit
1	1		add r1, 1 : r1			
2	0		add r8, 4 : r8	mul r5, r1 : r5		beq r1; r2 : ex
3	1		add r1, 1 : r1	... mul		
4	0	L:	add r8, 4 : r8	mul r5, r1 : r5	sto r5 : m r8	beq r1; r2 : ex
5	1		add r1, 1 : r1	... mul	... sto	bra L
6	1	ex:		... mul	... sto	
7	0				sto r5 : m r8	
8	1				... sto	
9	0					bra exit

prologue

software pipline with II = 2

epilogue

Lecture Compilation Methods SS 2013 / Slide 510

Objectives:

A software pipeline for a VLIW processor

In the lecture:

Explain

- the properties of the VLIW processor,
- the schedule,
- the software pipline,

Assignments:

- Make a table of run-times in cycles for n = 1, 2, ... iterations, and compare the figures without and with software pipelining.

5.2 / 6. Data Parallelism: Loop Parallelization

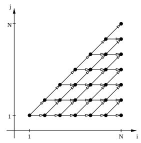
Regular loops on orthogonal data structures - parallelized for **data parallel** processors

Development steps (automated by compilers):

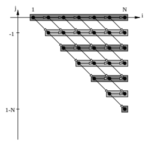
```
DECLARE B[0..N,0..N+1]

FOR I := 1 ..N
    FOR J := 1 .. I
        B[I,J] :=
            B[I-1,J]+B[I-1,J-1]
    END FOR
END FOR
```

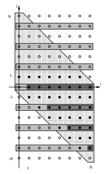
- **nested loops** operating on **arrays**,
  sequential execution of iteration space



- analyze **data dependences**
  data-flow: definition and use of array elements



- **transform loops**
  keep data dependences forward in time
- **parallelize inner loop(s)**
  map to field or vector of processors



- **map arrays to processors**
  such that many accesses are local,
  transform index spaces

Lecture Compilation Methods SS 2013 / Slide 511

Objectives:

Overview

In the lecture:

Explain

- Application area: scientific computations
- goals: execute inner loops in parallel with efficient data access
- transformation steps

Iteration space of loop nests

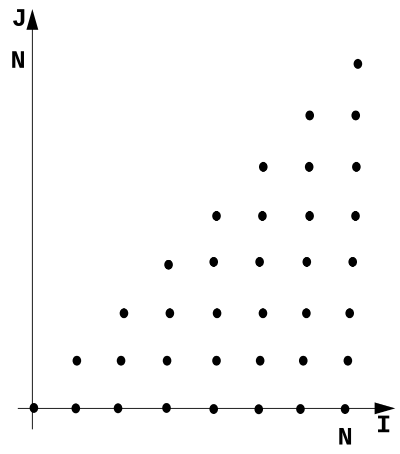
Iteration space of a loop nest of depth n:

- **n-dimensional space of integral points** (polytope)
- each point $(i_1, ..., i_n)$ represents an execution of the innermost loop body
- loop bounds are in general not known before run-time
- iteration need not have orthogonal borders
- iteration is elaborated sequentially

example:
computation of Pascal's triangle

```
DECLARE B[-1..N,-1..N]

FOR I := 0 .. N
   FOR J := 0 .. I
      B[I,J] :=
         B[I-1,J]+B[I-1,J-1]
   END FOR
END FOR
```



Lecture Compilation Methods SS 2013 / Slide 512

Objectives:

Understand the notion of iteration space

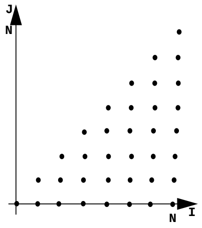
In the lecture:

- Explain the iteration space of the example.
- Show the order of elaboration of the iteration space.
- If the step size is greater than 1 the iteration space has gaps - the polytope is not convex.

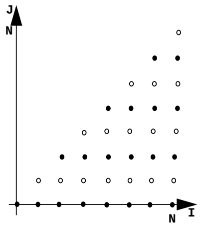
Questions:

- Draw an iteration space that has step size 3 in one dimension.

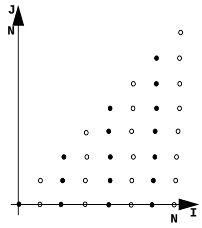
Examples for Iteration spaces of loop nests



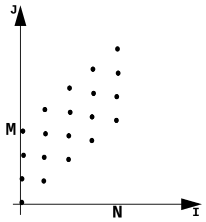

```
FOR I := 0 .. N
   FOR J := 0 .. I
```




```
FOR I := 0 .. N
   FOR J := 0..I BY 2
```

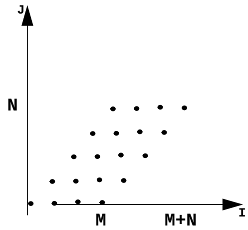



```
FOR I := 0..N BY 2
   FOR J := 0 .. I
```




```
FOR I := 0 .. N
   FOR J := I..I+M

M = 3, N = 4
```




```
FOR I := 0 .. M+N
   FOR J := max(0, I-M)..
            min (I, N)
```


Lecture Compilation Methods SS 2013 / Slide 512a

Objectives:

Relate loop nests to iteration spaces

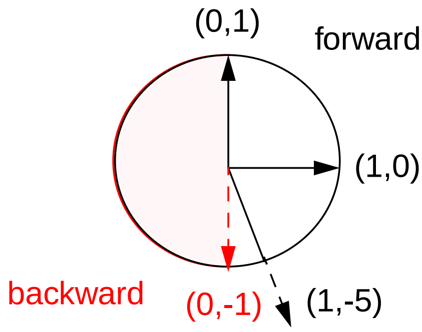
In the lecture:

- Explain the iteration spaces of the examples

Data Dependences in Iteration Spaces

Data dependence from iteration point i1 to i2:

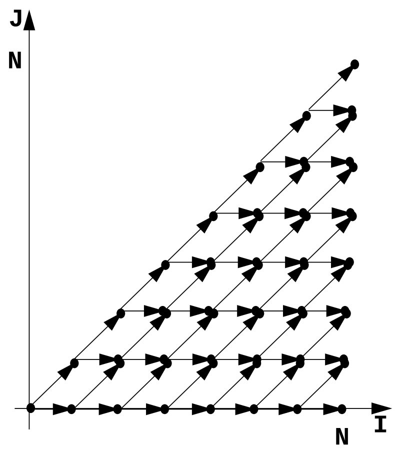
- Iteration **i**1 computes a value that is used in iteration **i2** (flow dependence)
- relative **dependence vector**
  d = **i2 - i1** = $(i2_1 - i1_1, ..., i2_n - i1_n)$
  holds for all iteration points except at the border
- Flow-dependences can **not be directed against the execution order**, can not point backward in time: each dependence vector must be **lexicographically positive**, i. e. **d** = $(0, ..., 0, d_i, ...)$, $d_i > 0$



Example:
Computation of Pascal´s triangle

```
DECLARE B[-1..N,-1..N]

FOR I := 0 .. N
   FOR J := 0 .. I
      B[I,J] :=
         B[I-1,J]+B[I-1,J-1]
   END FOR
END FOR
```



Lecture Compilation Methods SS 2013 / Slide 513

Objectives:

Understand dependences in loops

In the lecture:

Explain:

- Vector representation of dependences,
- examples,
- admissable directions graphically

Questions:

- Show different dependence vectors and array accesses in a loop body which cause dependences of given vectors.

Loop Transformation

The **iteration space** of a loop nest is transformed to **new coordinates**. Goals:

- **execute innermost loop(s) in parallel**
- improve **locality** of data accesses;
  in space: use storage of executing processor,
  in time: reuse values stored in cache
- **systolic** computation and communication scheme

Data dependences must **point forward in time**, i.e.
lexicographically positive and
not within parallel dimensions

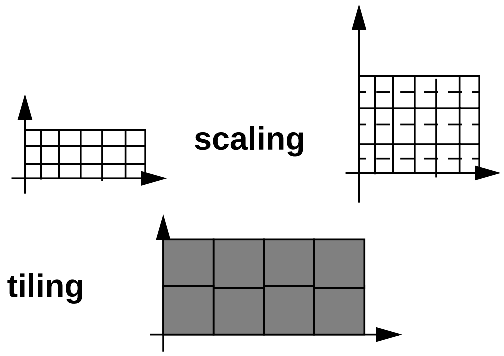
linear basic transformations:

- **Skewing**: add iteration count of an outer loop to that of an inner one
- **Reversal**: flip execution order for one dimension
- **Permutation**: exchange two loops of the loop nest

SRP transformations (next slides)

non-linear transformations, e. g.

- **Scaling**: **stretch the iteration space** in one dimension, causes gaps
- **Tiling**: introduce **additional inner loops** that **cover tiles** of fixed size



Lecture Compilation Methods SS 2013 / Slide 514

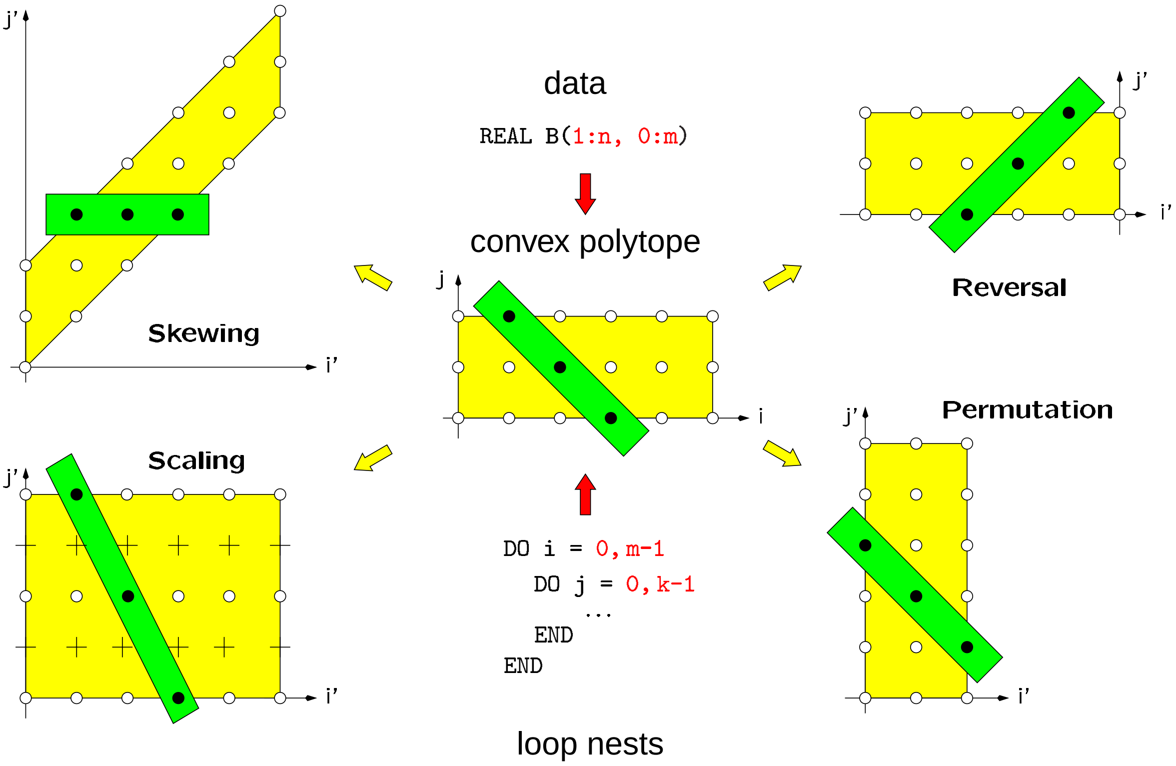
Objectives:

Overview

In the lecture:

- Explain the goals.
- Show admissable directions of dependences.
- Show diagrams for the transformations.

Transformations of



Lecture Compilation Methods SS 2013 / Slide 514a

Objectives:

Visualize the transformations

In the lecture:

- Give concrete loops for the diagrams.
- Show how the dependence vectors are transformed.
- Skewing and scaling do not change the order of execution; hence, they are always applicable.

Questions:

- Give dependence vectors for each transformation, which are still valid after the transformation.

Transformations defined by matrices

Transformation matrices: systematic transformation, check dependence vectors

Reversal $$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ -j \end{pmatrix} = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

Skewing $$\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ f*i+j \end{pmatrix} = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

Permutation $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} j \\ i \end{pmatrix} = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

Lecture Compilation Methods SS 2013 / Slide 514b

Objectives:

Understand the matrix representation

In the lecture:

- Explain the principle.
- Map concrete iteration points.
- Map dependence vectors.
- Show combinations of transformations.

Questions:

- Give more examples for skewing transformations.

Reversal

Iteration count of one loop is negated, that dimension is enumerated backward

general transformation matrix

$$\begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \mathbf{-1} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$$

2-dimensional:

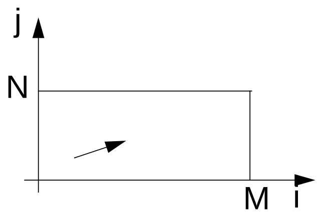
loop variables: old, new

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ -j \end{pmatrix} = \begin{pmatrix} ir \\ jr \end{pmatrix}$$

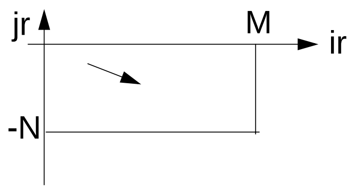
```
for i = 0 to M
   for j = 0 to N
      ...
```

```
for ir = 0 to M
   for jr = -N to 0
      ...
```

original



transformed



Lecture Compilation Methods SS 2013 / Slide 515

Objectives:

Understand reversal transformation

In the lecture:

- Explain the effect of reversal transformation.
- Explain the notation of the transformation matrix.
- There may be no dependences in the direction of the reversed loop - they would point backward after the transformation.

Questions:

- Show an example where reversal enables loop fusion.

Skewing

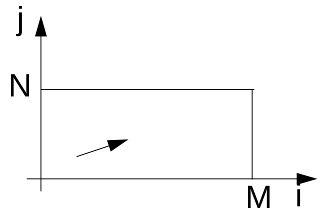
The **iteration count** of an outer loop is **added to the count of an inner loop;** iteration space is shifted; **execution order** of iteration points **remains unchanged**

general transformation matrix:

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & \mathbf{f} & 1 & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{pmatrix}$$

```
for i = 0 to M
   for j = 0 to N
      ...
```

original



2-dimensional:

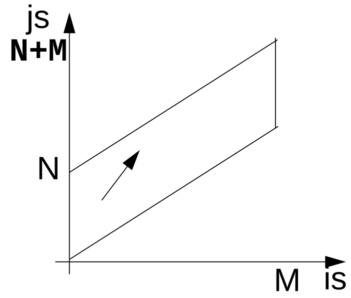
loop variables

old new

$$\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ f*i+j \end{pmatrix} = \begin{pmatrix} is \\ js \end{pmatrix}$$

```
for is = 0 to M
   for js = f*is to N+f*is
      ...
```

transformed



Lecture Compilation Methods SS 2013 / Slide 516

Objectives:

Understand skewing transformation

In the lecture:

- Explain the effect of a skewing transformation.
- Skewing is always applicable.
- Skewing can enable loop permutation

Questions:

- Show an example where skewing enables loop permutation.

Permutation

Two loops of the loop nest are interchanged; the iteration space is flipped;
the **execution order** of iteration points **changes;** new dependence vectors must be legal.

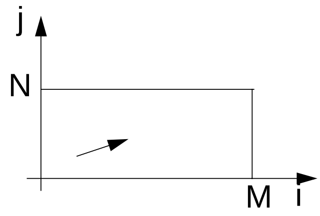
general transformation matrix:

$$\begin{array}{c} \\ i \\ j \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & & \\ & \mathbf{0} & \mathbf{1} & & & 0 \\ & & 1 & & & \\ & \mathbf{1} & \mathbf{0} & & & \\ 0 & & & 1 & \dots & \\ & & & & & 1 \end{pmatrix}$$

$$\quad\quad\quad i \quad j$$

```
for i = 0 to M
   for j = 0 to N
      ...
```

original



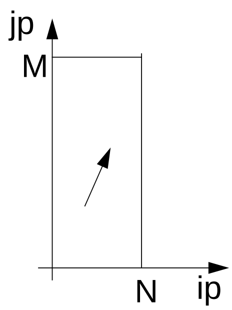
2-dimensional:

loop variables
old new

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} j \\ i \end{pmatrix} = \begin{pmatrix} ip \\ jp \end{pmatrix}$$

```
for ip = 0 to N
   for jp = 0 to M
      ...
```

transformed



Lecture Compilation Methods SS 2013 / Slide 517

Objectives:

Understand loop permutation

In the lecture:

- Explain the effect of loop permutation.
- Show effect on dependence vectors.
- Permutation often yields a parallelizable innermost loop.

Questions:

- Show an example where permutation yields a parallelizable innermost loop.

Use of Transformation Matrices

- Transformation matrix **T** defines **new iteration counts** in terms of the old ones: **T * i = i´**

e. g. Reversal

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ -j \end{pmatrix} = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

- Transformation matrix **T** transforms old **dependence vectors** into new ones: **T * d = d´**

e. g.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- inverse Transformation matrix **T** $^{-1}$ defines **old iteration counts** in terms of new ones, for transformation of index expressions in the loop body: $\mathbf{T^{-1} * i' = i}$

e. g.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{pmatrix} i' \\ -j' \end{pmatrix} = \begin{pmatrix} i \\ j \end{pmatrix}$$

- **concatenation of transformations** first $\mathbf{T_1}$ then $\mathbf{T_2}$: $\mathbf{T_2 * T_1 = T}$

e. g.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Lecture Compilation Methods SS 2013 / Slide 518

Objectives:

Learn to Use the matrices

In the lecture:

- Explain the 4 uses with examples.
- Transform a loop completely.

Questions:

- Why do the dependence vectors change under a transformation, although the dependence between array elements remains unchanged?

Inequalities Describe Loop Bounds

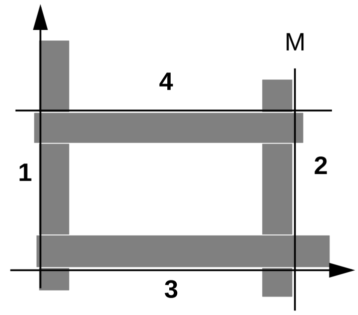
The bounds of a loop nest are described by a **set of linear inequalities**.
Each **inequality separates the space** in „inside and outside of the iteration space“:

$$\mathbf{B * i \le c}$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} \le \begin{pmatrix} 0 \\ M \\ 0 \\ N \end{pmatrix}$$

example 1

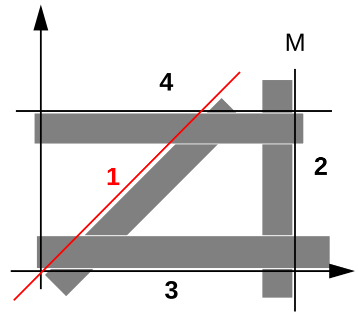
1 **-i** ≤ 0
2 **i** $\le M$
3 **-j** ≤ 0
4 **j** $\le N$



$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} i \\ j \end{pmatrix} \le \begin{pmatrix} 0 \\ M \\ 0 \\ N \end{pmatrix}$$

example 2

1 **-i +j** ≤ 0
2 **i** $\le M$
3 **-j** ≤ 0
4 **j** $\le N$



transformed

positive factors represent **upper** bounds
negative factors represent **lower** bounds

1, 4: $j \le \min(i, N)$ **1+ 3:** $0 \le i$
3: $0 \le j$ **2:** $i \le M$

Lecture Compilation Methods SS 2013 / Slide 519

Objectives:

Understand representation of bounds

In the lecture:

- Explain matrix notation.
- Explain graphic interpretation.
- There can be arbitrary many inequalities.

Questions:

- Give the representations of other iteration spaces.

Transformation of Loop Bounds

The inverse of a transformation matrix $\mathbf{T}^{-1}$ transforms a set of inequalities: $\mathbf{B} * \mathbf{T}^{-1}\, \mathbf{i'} \leq \mathbf{c}$

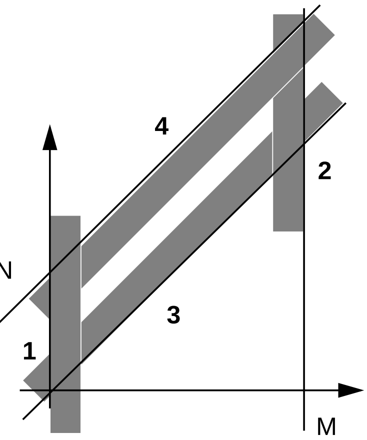
skewing $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ inverse $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}}_{B} * \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{T^{-1}} = \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}}_{B * T^{-1}}$$

example 1
new bounds:

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}}_{B * T^{-1}} * \underbrace{\begin{pmatrix} i' \\ j' \end{pmatrix}}_{i'} \leq \underbrace{\begin{pmatrix} 0 \\ M \\ 0 \\ N \end{pmatrix}}_{c}$$

1 $-i' \leq 0$
2 $i' \leq M$
3 $i' - j' \leq 0$
4 $-i' + j' \leq N$



Lecture Compilation Methods SS 2013 / Slide 520

Objectives:

Understand the transformation of bounds

In the lecture:

- Explain how the inequalities are transformed

Questions:

- Compute further transformations of bounds.

Example for Transformation and Parallelization of a Loop

```
for i = 0 to N
   for j = 0 to M
      a[i, j] = (a[i, j-1] + a[i-1, j]) / 2;
```

Parallelize the above loop.

1. Draw the iteration space.
2. Compute the dependence vectors and draw examples of them into the iteration space. Why can the inner loop not be executed in parallel?
3. Apply a skewing transformation and draw the iteration space.
4. Apply a permutation transformation and draw the iteration space. Explain why the inner loop now can be executed in parallel.
5. Compute the matrix of the composed transformation and use it to transform the dependence vectors.
6. Compute the inverse of the transformation matrix and use it to transform the index expressions.
7. Specify the loop bounds by inequalities and transform them by the inverse of the transformation matrix.
8. Write the complete loops with new loop variables ip and jp and new loop bounds.

Lecture Compilation Methods SS 2013 / Slide 521

Objectives:

Exercise the method for an example

In the lecture:

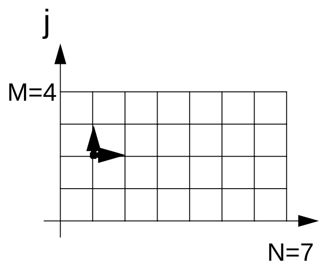
- Explain the steps of the transformation.
- Solution on C-5.22

Questions:

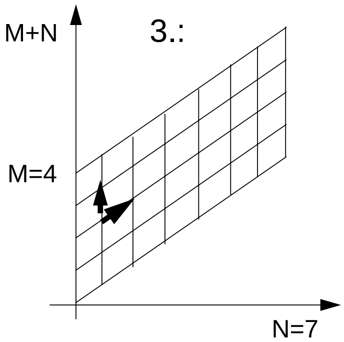
- Are there other transformations that lead to a parallel inner loop?

Solution of the Transformation and Parallelization Example

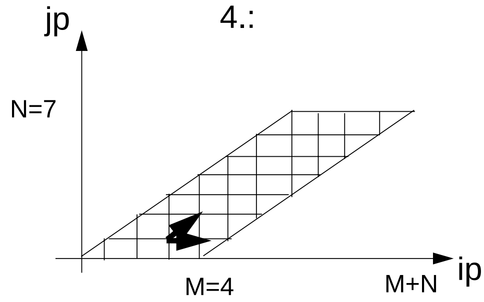
1., 2.:



3.:



4.:



5.: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

6.: Inverse $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

7. Bounds:

orig.: B $\begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$ c $\begin{pmatrix} 0 \\ N \\ 0 \\ M \end{pmatrix}$

new: $B * T^{-1}$ $\begin{pmatrix} 0 & -1 \\ 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$

1 $-jp \le 0$
2 $jp \le N$
3 $-ip+jp \le 0$
4 $ip - jp \le M$

1, 3 => $0 \le ip$
2, 4 => $ip \le M+N$
1, 4 => $\max(0, ip-M) \le jp$
2, 3 => $jp \le \min(ip, N)$

8.
```
for ip = 0 to M+N
    for jp = max (0, ip-M) to min (ip, N)
        a[jp, ip-jp] = (a[jp, ip-jp-1] + a[jp-1, ip-jp]) / 2;
```


Lecture Compilation Methods SS 2013 / Slide 522

Objectives:

Solution for C-60

In the lecture:

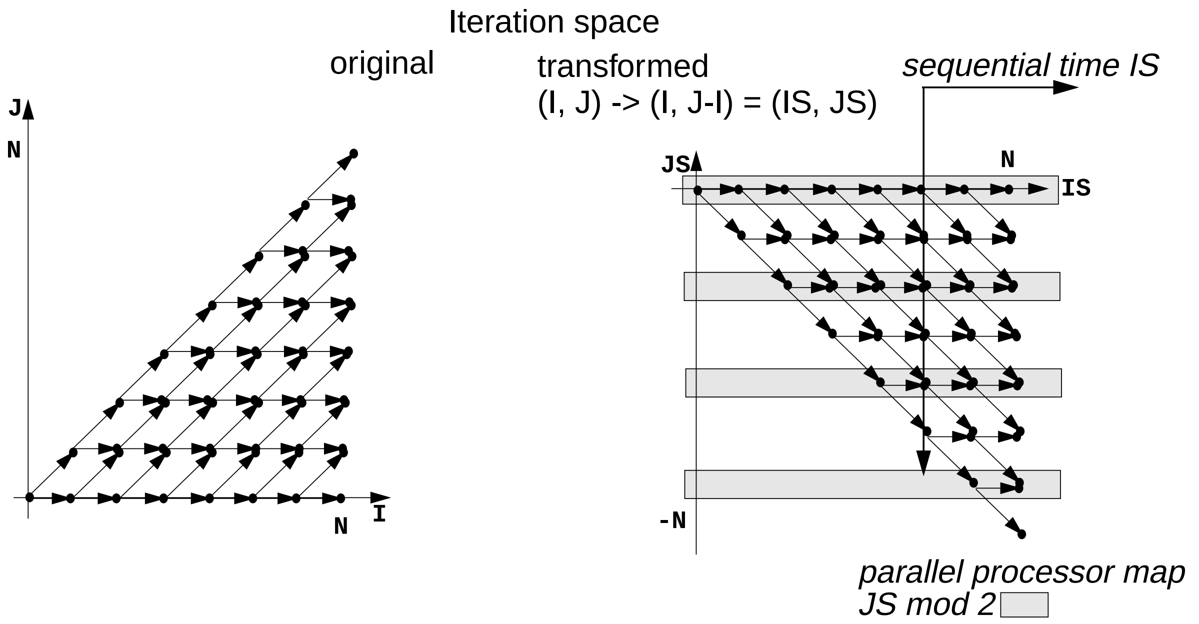
Explain

- the bounds of the iteration spaces,
- the dependence vectors,
- the transformation matrix and its inverse,
- the conditions for being parallelizable,
- the transformation of the index expressions
- the transformation of the loop bounds.

Questions:

- Describe the transformation steps.

Transformation and Parallelization




```
DECLARE B[-1..N,-1..N]

FOR I := 0 .. N
   FOR J := 0 .. I
      B[I,J] :=
         B[I-1,J]+B[I-1,J-1]
   END FOR
END FOR
```

```
DECLARE B[-1..N,-1..N]

FOR IS := 0.. N
   FOR JS := -IS .. 0
      B[IS,JS+IS] :=
         B[IS-1,JS+IS]+B[IS-1,JS-1+IS]
   END FOR
END FOR
```

Lecture Compilation Methods SS 2013 / Slide 523

Objectives:

Example for parallelization

In the lecture:

- Explain skewing transformation: f = -1
- Inner loop in parallel.
- Explain the time and processor mapping.
- mod 2 folds the arbitrary large loop dimension on a fixed number of 2 processors.

Questions:

- Give the matrix of this transformation.
- Use it to compute the dependence vectors, the index expressions, and the loop bounds.

Data Mapping

Goal:

Distribute array elements over processors, such that as many **accesses as possible are local.**

Index space of an array:

n-dimensional space of integral index points (polytope)

- **same properties as iteration space**
- same mathematical model
- same **transformations** are applicable (Skewing, Reversal, Permutation, ...)
- **no restrictions** by data dependences

Lecture Compilation Methods SS 2013 / Slide 524

Objectives:

Reuse model of iteration spaces

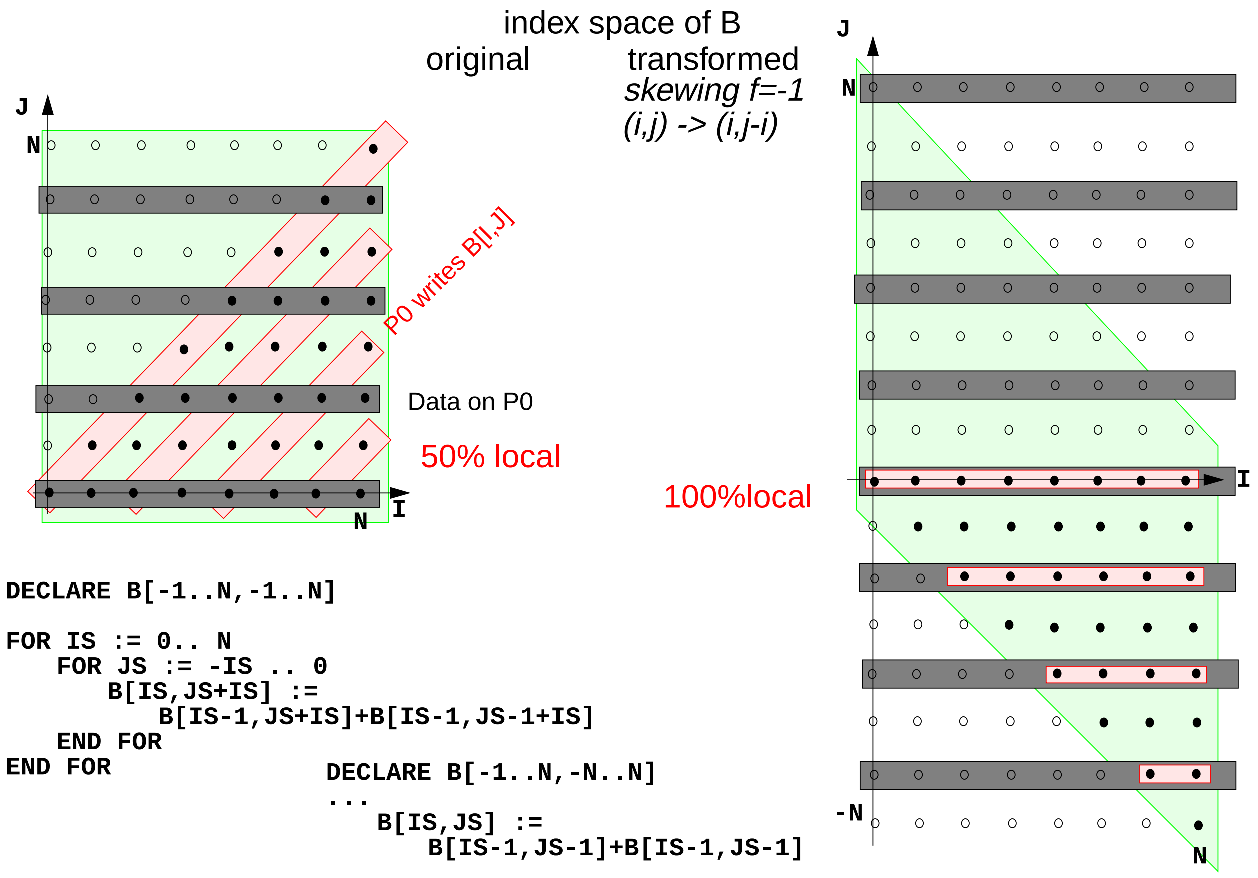
In the lecture:

Explain, using examples of index spaces

Questions:

- Draw an index space for each of the 3 transformations.

Data distribution for parallel loops



Lecture Compilation Methods SS 2013 / Slide 525

Objectives:

The gain of an index transformation

In the lecture:

Explain

- local and non-local accesses,
- the index transformation,
- the gain of locality,
- unused memory because of skewing.

Questions:

- How do you compute the index transformation using a transformation matrix?